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There is No Axiomatic System for the Quantum Theory

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Abstract We show that there is a contradiction within quantum mechanics. We derive a proposition concerning a quantum expectation value under the assumption of the existence of the directions in a spin-1/2 system. The quantum predictions within the formalism of von Neumann's projective measurement cannot coexist with the proposition concerning the existence of the directions. Therefore, we have to give up either the existence of the directions or the formalism of von Neumann's projective measurement. Hence there is a contradiction within the Hilbert space formalism of the quantum theory. This implies that there is no axiomatic system for the quantum theory. We need new physical theories in order to explain mathematically the handing of raw experimental data.

Keywords Quantum measurement theory · Quantum computer

1 Introduction

As a famous physical theory, the quantum theory (cf. [1–6]) gives accurate and at times remarkably accurate numerical predictions. Much experimental data fits to the quantum predictions for the past some 100 years. The quantum theory also says new science with respect to information theory. The science is called the quantum information theory [6]. Therefore, the quantum theory gives us very useful another theory in order to create a new information science and to explain the handing of raw experimental data.

As for the foundations of the quantum theory, Leggett-type nonlocal variables theory [7] is experimentally investigated [8–10]. The experiments report that the quantum theory does not accept Leggett-type nonlocal variables interpretation. As for the applications of the quantum theory, there are several attempts to use single-photon two-qubit states for quantum computing. Oliveira *et al.* implement Deutsch's algorithm [11] with polarization and transverse spatial modes of the electromagnetic field as qubits [12]. Single-photon Bell states are prepared and measured [13]. Also the decoherence-free implementation of Deutsch's

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algorithm is reported by using such single-photon and by using two logical qubits [14]. More recently, a one-way based experimental implementation of Deutsch's algorithm is reported [15].

To date, the quantum theory seems to be a successful physical theory and it looks to have no problem in order to use it experimentally. Several researches address [1] the mathematical formulation of the quantum theory. It is desirable that the quantum theory is also mathematically successful because we predict unknown physical phenomena precisely. Sometimes such predictions are effective in the field of elementary particle physics. We endure much time in order to see the fact by using, for example, large-scale accelerator. Further, Rolf Landauer says that *Information is Physical* [6]. We cannot create any computer without physical phenomena. This fact motivates us to investigate the Hilbert space formalism of the quantum theory.

Here we aim to show that there is a contradiction within the Hilbert space formalism of the quantum theory. We know that a theory means a set of propositions. Unfortunately, we have to abandon that the quantum theory satisfies consistency, which is necessary in order to have axiomatic system. This implies that there is no axiomatic system for the quantum theory. A theory K may be said to be consistent if any proposition, $A \in K$, belonging to the theory K and the negation of the proposition, $A^{\neg}$, are not derived, simultaneously. Otherwise, the theory K may be said to be contradictory. Our discussion says that, surprisingly, the quantum theory is a contradictory physical theory in order to explain mathematically the handing of raw experimental data.

Our discussion is very important. We need new physical theories in order to explain raw data informationally, to create new information science, and to predict new unknown physical phenomena efficiently. What are new physical theories? We cannot answer it at this stage. However, we expect that our discussion in this thesis could contribute to creating new physical theories in order to explain the handing of raw experimental data, to create new information science, and to predict new unknown physical phenomena efficiently.

Our thesis is organized as follows. We derive a proposition concerning a quantum expectation value under the assumption of the existence of the directions in a spin-1/2 system. The quantum predictions within the formalism of von Neumann's projective measurement (the results of measurements are ± 1) cannot coexist with the proposition concerning the existence of the directions. Therefore, there is a contradiction in the set of propositions of the quantum theory in a spin-1/2 system, viz., there is no axiomatic system for the quantum theory. What we need is only one pure spin-1/2 state lying in the x-y plane (a two-dimensional state).

Throughout this thesis, we confine ourselves to the two-level (e.g., electron spin, photon polarizations, and so on) and the discrete eigenvalue case. The number of settings of measuring apparatuses is two (two-setting model). These assumptions are used in several experimental situations.

2 There Is a Contradiction within Quantum Mechanics

2.1 The Existence of the Directions

In what follows, we show that there is a contradiction within quantum mechanics. Assume a pure spin-1/2 state ψ lying in the x-y plane. Let $\vec{\sigma}$ be $(\sigma_x, \sigma_y, \sigma_z)$, the vector of Pauli operators. The measurements (observables) on a spin-1/2 state lying in the x-y plane of $\vec{n} \cdot \vec{\sigma}$ are parameterized by a unit vector $\vec{n}$ (its direction along which the spin component is measured). Here, $\cdot$ is the scalar product in $\mathbf{R}^3$.

We have a quantum expectation value $E_{\text{QM}}^k,\ k=1,2$ as

$$E_{\text{QM}}^k \equiv \text{Tr}[\psi \vec{n}_k \cdot \vec{\sigma}], \quad k=1,2. \tag{1}$$

We have $\vec{x} \equiv \vec{x}^{(1)}$, $\vec{y} \equiv \vec{x}^{(2)}$, and $\vec{z} \equiv \vec{x}^{(3)}$ which are the Cartesian axes relative to which spherical angles are measured. Let us write the two unit vectors in the plane defined by $\vec{x}^{(1)}$ and $\vec{x}^{(2)}$ in the following way:

$$\vec{n}_k = \cos\theta_k \vec{x}^{(1)} + \sin\theta_k \vec{x}^{(2)}. \tag{2}$$

Here, the angle θ_k takes only two values: $\theta_1 = 0,\ \theta_2 = \frac{\pi}{2}$.

We derive a necessary condition for the quantum expectation value for the system in a pure spin-1/2 state lying in the x-y plane given in (1). We derive the possible values of the scalar product $\sum_{k=1}^{2}(E_{\text{QM}}^k \times E_{\text{QM}}^k) \equiv \|E_{\text{QM}}\|^2$. E_{QM}^k is the quantum expectation value given in (1). We see that $\|E_{\text{QM}}\|^2 = \langle\sigma_x\rangle^2 + \langle\sigma_y\rangle^2$. We use decomposition (2). We introduce simplified notations as $T_i = \text{Tr}[\psi \vec{x}^{(i)} \cdot \vec{\sigma}]$ and $(c_k^1, c_k^2) = (\cos\theta_k, \sin\theta_k)$. Then, we have

$$\|E_{\text{QM}}\|^2 = \sum_{k=1}^{2}\left(\sum_{i=1}^{2} T_i c_k^i\right)^2 = \sum_{i=1}^{2} T_i^2 \leq 1, \tag{3}$$

where we use the orthogonality relation $\sum_{k=1}^{2} c_k^\alpha c_k^\beta = \delta_{\alpha,\beta}$. From a proposition of the quantum theory, the Bloch sphere (the directions) with the value of $\sum_{i=1}^{2} T_i^2$ is bounded as $\sum_{i=1}^{2} T_i^2 \leq 1$. The reason of the condition (3) is the Bloch sphere $\sum_{i=1}^{3}(\text{Tr}[\psi \vec{x}^{(i)} \cdot \vec{\sigma}])^2 \leq 1$. Thus we derive a proposition concerning a quantum expectation value under the assumption of the existence of the directions (in a spin-1/2 system), that is, $\|E_{\text{QM}}\|^2 \leq 1$. It is worth noting here that this inequality must be saturated if ψ is a pure state lying in the x-y plane. That is, $\sum_{i=1}^{2}(\text{Tr}[\psi \vec{x}^{(i)} \cdot \vec{\sigma}])^2 = 1$. Hence we derive the following proposition concerning the existence of the directions when the system is in a pure state lying in the x-y plane

$$\|E_{\text{QM}}\|_{\max}^2 = 1. \tag{4}$$

$\|E_{\text{QM}}\|_{\max}^2$ is the maximal possible value of the scalar product.

2.2 von Neumann's Projective Measurement

On the other hand, let us assume von Neumann's projective measurement. In this case, the quantum expectation value in (1), which is the average of the results of projective measurements, is given by

$$E_{\text{QM}}^k = \lim_{m\to\infty} \frac{\sum_{l=1}^{m} r_l(\vec{n}_k)}{m}. \tag{5}$$

The possible values of the actually measured result $r_l(\vec{n}_k)$ are ± 1 (in $\hbar/2$ unit). Same quantum expectation value is given by

$$E_{\text{QM}}^k = \lim_{m'\to\infty} \frac{\sum_{l'=1}^{m'} r_{l'}(\vec{n}_k)}{m'}, \tag{6}$$

because we only change the labels as $m \to m'$ and $l \to l'$. Of course, the possible values of the actually measured result $r_{l'}(\vec{n}_k)$ are ± 1 (in $\hbar/2$ unit). Thus, we have

$$\{l \mid l \in \mathbf{N} \wedge r_l(\vec{n}_k) = 1\} = \{l' \mid l' \in \mathbf{N} \wedge r_{l'}(\vec{n}_k) = 1\} \tag{7}$$

and

$$\{l|l \in \mathbf{N} \wedge r_l(\vec{n}_k) = -1\} = \{l'|l' \in \mathbf{N} \wedge r_{l'}(\vec{n}_k) = -1\}. \tag{8}$$

By using these facts, we derive a necessary condition for the quantum expectation value for the system in a pure spin-1/2 state lying in the x-y plane given in (5). Again, we derive the possible values of the scalar product $\|E_{\text{QM}}\|^2$ of the quantum expectation value, E_{QM}^k given in (5). We have

$$\begin{aligned}
\|E_{\text{QM}}\|^2 &= \sum_{k=1}^{2} \left(\lim_{m\to\infty} \frac{\sum_{l=1}^{m} r_l(\vec{n}_k)}{m} \times \lim_{m'\to\infty} \frac{\sum_{l'=1}^{m'} r_{l'}(\vec{n}_k)}{m'} \right) \\
&= \sum_{k=1}^{2} \left(\lim_{m\to\infty} \frac{\sum_{l=1}^{m}}{m} \cdot \lim_{m'\to\infty} \frac{\sum_{l'=1}^{m'}}{m'} r_l(\vec{n}_k) r_{l'}(\vec{n}_k) \right) \\
&\leq \sum_{k=1}^{2} \left(\lim_{m\to\infty} \frac{\sum_{l=1}^{m}}{m} \cdot \lim_{m'\to\infty} \frac{\sum_{l'=1}^{m'}}{m'} |r_l(\vec{n}_k) r_{l'}(\vec{n}_k)| \right) \\
&= \sum_{k=1}^{2} \left(\lim_{m\to\infty} \frac{\sum_{l=1}^{m}}{m} \cdot \lim_{m'\to\infty} \frac{\sum_{l'=1}^{m'}}{m'} \right) = 2.
\end{aligned} \tag{9}$$

The above inequality can be saturated since, as we have said,

$$\{l|l \in \mathbf{N} \wedge r_l(\vec{n}_k) = 1\} = \{l'|l' \in \mathbf{N} \wedge r_{l'}(\vec{n}_k) = 1\} \tag{10}$$

and

$$\{l|l \in \mathbf{N} \wedge r_l(\vec{n}_k) = -1\} = \{l'|l' \in \mathbf{N} \wedge r_{l'}(\vec{n}_k) = -1\}. \tag{11}$$

Thus we derive a proposition concerning a quantum expectation value under the assumption that von Neumann's projective measurement is true (in a spin-1/2 system), that is, $\|E_{\text{QM}}\|^2 \leq 2$. Hence we derive the following proposition concerning von Neumann's projective measurement

$$\|E_{\text{QM}}\|^2_{\max} = 2. \tag{12}$$

Clearly, we cannot assign the truth value "1" for two propositions (4) (concerning the existence of the directions) and (12) (concerning von Neumann's projective measurement), simultaneously, when the system is in a pure state lying in the x-y plane. Therefore, we are in the contradiction when the system is in a pure state lying in the x-y plane.

3 Conclusions

In conclusion, we have shown that there is a contradiction within quantum mechanics. The quantum predictions within the formalism of von Neumann's projective measurement cannot have coexisted with the existence of the directions. These quantum-theoretical propositions have been contradicted each other. Therefore there has been a contradiction in the set of propositions of the quantum theory. Hence there has been no axiomatic system for the quantum theory. Our discussion has been obtained in a quantum system which is in a pure spin-1/2 state lying in the x-y plane.

What are new physical theories? We cannot answer it at this stage. However, we expect that our discussion in this thesis could contribute to creating new physical theories in order to explain the handing of raw experimental data, to create new information science, and to predict new unknown physical phenomena efficiently.

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